Signal Propagation in a Positive Definite Riemannian Space

C. LANCZOS
Dublin Institute for Advanced Studies, Dublin, Ireland
(Received 7 November 1963)

It is commonly accepted that the propagation of signals requires a space which is hyperbolic in time. The indefiniteness of the metric thus established contradicts the natural requirements of a rational metric. The proof is given that a genuinely Riemannian (positive definite) space of fourfold lattice structure is well suited to the propagation of signals, if the g_{ik} assume very large values along some narrow ridge surfaces. The resulting signal propagation is strictly translational and has the nature of a particle which moves with light velocity (photon). According to this theory the discrepancy between classical and quantum phenomena is caused by the misinterpretation of a Riemannian metric in Minkowskian terms. The Minkowskian metric comes about (in high approximation) macroscopically, in dimensions which are large in comparison to the fundamental lattice constant. Since this constant is of the order 10^{-32} cm, this condition is physically always fulfilled.

I. INTRODUCTION

IT is commonly assumed that the propagation of signals with light velocity demands a D'Alembert type of operator (elliptic in space, hyperbolic in time) which establishes a fundamental difference in the manner in which the space coordinates and time enter the field equations of mathematical physics. The D'Alembert operator can be conceived as the potential operator of a space which has the Minkowskian metric $g_{ik}=\eta_{ik}$, where $\eta_{ik}=0$ for $i\neq k$, and $=-1,\ -1,\ -1,\ +1$ for $i=k=1,\ 2,\ 3,\ 4$. From the standpoint of geometry the indefinite character of the Minkowskian metric is a severe handicap, because it violates the two fundamental principles of a rational metric:

$$(1)\quad AB=0 \text{ implies } A=B\,, \tag{1}$$

$$(2)\qquad AC\leq AB+BC\,. \tag{2}$$

The first principle is violated because any two points of a Minkowskian space can be connected by lines of zero length, and in infinitely many ways. The second principle is violated, because in a metric which is not positive definite, the distance between two points cannot be defined as the shortest distance between two points, but merely as the stationary value of the integral of ds between definite limits. Nor is a consistent differential geometry possible in a space of this structure. Every point of a Minkowskian manifold is the center of an island universe (limited by the null cone) and the relation between these universes is established with the help of the Minkowskian coordinates (in which the g_{ik} become η_{ik}) which, however, have no invariant significance.

Einstein[1] was not in favor of the geometrical language which is, in fact, void of significance if we perform operations of a purely analytical nature and label them by geometrical names, although in truth the concepts of geometry are not applicable to them. (He considered the minus sign of the metric as an unsolved riddle of physics which may find its explanation at some future date.) For him the theory of relativity appeared essentially as the study of tensors and their transformations, associated with a certain invariant differential form of the second order.

In a recent paper,[2] the author endeavored to show that a truly Riemannian space of the positive definite signature $++++$ [which satisfies the two postulates (1) and (2)], can nevertheless in *macroscopic* relations simulate the behavior of a Minkowskian line element. Since this Riemannian space is highly curved, we must be able to explain why these large local curvatures cancel out under macroscopic conditions, thus giving the impression of a flat pseudo-Euclidean manifold. The explanation was found in the picture of a space-time world of *fourfold periodicity*, with a lattice constant which is immensely small. This results in the picture of a *fundamental metrical cell*, whose metric is identically repeated throughout the entire space-time world. This is so far the picture of the empty (particleless) universe. The material particles have to be conceived as nonperiodic modifications of the basic periodic metric, extending over a very large number of cells (and influencing to a minor degree the metric throughout the space-time world).

The idea of Euclidean operators for the purposes of quantum theory was introduced by Nakano[3] and, independently, by Schwinger.[4] Treder[5] considers the possibility of a metric which changes from elliptic (in nucleonic domains) to hyperbolic (in outside space), although this transition demands the vanishing of the metrical determinant on the boundary between the two regions. The author's discussion is restricted to classical geometrical field equations in the sense of Einstein (although abandoning the field equations $R_{ik}=0$ which Einstein himself did not consider as of final significance) but dropping the quasiflat hypothesis and replacing it by a strongly curved periodic Riemannian field which, however, becomes purely Euclidean in infinitesimal

[1] A. Einstein, *Philosopher-Scientist* (Library of Living Philosophers, Evanston, Illinois, 1949), p. 61.

[2] C. Lanczos, J. Math. Phys. **4**, 951 (1963).
[3] T. Nakano, Progr. Theoret. (Kyoto) Phys. **21**, 241 (1959).
[4] J. Schwinger, Phys. Rev. **115**, 721 (1959).
[5] H. Treder, Ann. Phys. (Paris) **9**, 284 (1962).

dimensions (in harmony with any genuinely Riemannian geometry).

The derivation of the results in the quoted paper[2] occurred with the help of a few auxiliary assumptions which did not follow from the basic postulates. It is the purpose of the present paper to demonstrate that the assumption of a positive definite Riemannian space with a fourfold lattice structure is sufficient for the propagation of signals—usually considered to be the privilege of a space which is hyperbolic in the fourth dimension.

II. HIGH RIDGES OF THE METRICAL TENSOR

We start with the classical problem of potential theory, called the "Dirichlet problem." We want to minimize the action integral which has the Lagrangian density

$$|\mathrm{grad}\ \varphi|^2 \tag{3}$$

prescribing the values of φ on the boundary σ of our domain τ. However, we want to add the restricting condition that only such functions are admitted as test functions whose normal derivative $\partial\varphi/\partial n$ (to be denoted by $\partial_n\varphi$) is zero on the boundary. This is not permitted under ordinary circumstances, since we cannot prescribe both φ *and* $\partial_n\varphi$ on the boundary. We can, however, consider our variational problem with the added *auxiliary condition* $\partial_n\varphi=0$ on σ, (with the realization that we are not going to obtain the usual $\Delta\varphi=0$ as the solution of our problem). We replace the local condition "$\partial_n\varphi=0$ on σ" by the single global condition

$$\int\left(\frac{\partial\varphi}{\partial n}\right)^2 d\sigma=0\,, \tag{4}$$

which in fact has the consequence that $\partial_n\varphi$ must vanish at every point of σ. By treating the condition (4) by the usual Lagrangian multiplier method, we obtain the modified action integral

$$A'=\int|\mathrm{grad}\,\varphi|^2 d\tau+\lambda\int\left(\frac{\partial\varphi}{\partial n}\right)^2 d\sigma. \tag{5}$$

Here λ is a large positive constant (or possibly a function of σ) which tends to infinity. We now change the second integral to a space integral with the help of Dirac's delta function:

$$A'=\int\left[|\mathrm{grad}\,\varphi|^2+\lambda\delta(\tau,\sigma)\left(\frac{\partial\varphi}{\partial n}\right)^2\right]d\tau. \tag{6}$$

Now we smear the delta function over a narrow strip of the width ϵ around σ, thus making it a regular space function. This function is zero everywhere, except in a narrow strip around σ, where it increases to very high values. Furthermore, $\lambda(\tau)$ should not grow to infinity but merely be very large. Under these circumstances

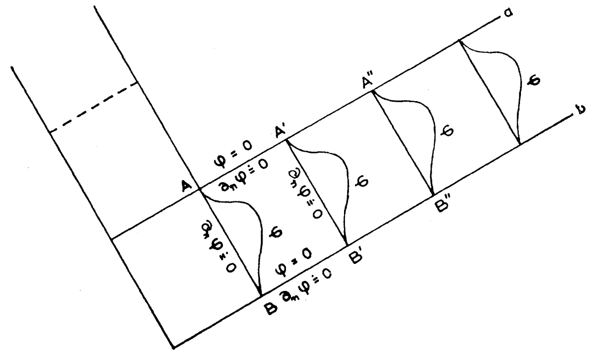

FIG. 1. Signal propagation by hedge-hopping.

the second term has the following effect. Since the coefficients of $\partial\varphi/\partial x_i$ are so much larger in the second term than in the first term, a good minimum automatically demands that $\partial_n\varphi$ must remain *very small* (although not zero) all along σ.

The action integral (6) can be interpreted as the usual action integral of the potential operator of a properly defined (positive definite) Riemannian space. This space has a peculiar metric. In every elementary cell the g_{ik} have almost everywhere the usual Euclidean values, except along some narrow ridges, where they climb to very high values, as it was discussed in the author's previously quoted paper.[6]

Let us first investigate the problem in two dimensions. In Fig. 1 we see the two edges AB and $A'B'$ along which the function $\partial_n\varphi$ must remain practically zero at all points. We will prescribe $\varphi(x_1,x_2)$ along the two edges in such a way that the functional values shall be the same in corresponding points of the two edges: $\varphi(x_1,x_2)=\varphi(x_1',x_2')$. Now we can argue as follows. For the moment we want to assume that $\lambda(\tau)$ becomes in fact *infinite* along these edges. Then $\partial_n\varphi$ vanishes along both edges and since now both φ and $\partial_n\varphi$ are the same in corresponding points, and the cell structure is strictly identical throughout space, the rectangle $AA'BB'$ can be bodily transplanted to $A'A''B'B''$, and so on, forever. In physical interpretation we have obtained a solution of the potential equation of a Riemannian space, in which a "signal" propagates to the right (or to the left) with light velocity, without attenuation. Outside the infinite channel enclosed by the two straight lines a and b, the function $\varphi(x_1,x_2)$ vanishes; (since both φ and $\partial_n\varphi$ are zero on the boundary). The propagated signal remains constant only macroscopically because $\varphi(x_1,x_2)$ is *not* constant *between* the two edges; (these two edges need not belong to neighboring cells but to two cells separated by any number of fundamental cells between). We get fluctuations between the ridges but they are of such high frequency that their effect remains unobservable. The event may be described as "hedge-hopping," but the two edges are too near to each other

[6] See, in particular, p. 954.

to cause a macroscopically observable deviation from a steady propagation of the signal.

Obviously we have here a principle which is applicable to a much wider class of phenomena. The ridge lines can be replaced by ridge surfaces and the infinite channel between two parallel ridges by an infinite rectangular trough, which will contain the entire radiation. If we find two corresponding ridge surfaces, on which $\varphi(x_i)=\varphi(x_i')$ in corresponding points, then the vanishing of $\partial_n\varphi$ (due to the geometry of space) adds the condition $\partial_n\varphi(x_i)=\partial_n\varphi(x_i')$ in corresponding points. Then the lattice structure of space has the consequence that the signal is propagated by hedge hopping forever. Of decisive significance here is the fact that the strict periodicity of the repetition excludes any shrinking or stretching of the signal. The propagation occurs by strict *parallel translation*, without any scale transformation (such as demanded by the customary $1/r$ law). We have the complete picture of Einstein's "needle radiation,"[7] demanded by the localized nature of the photon. The signal emission induced by our space structure has the character of a particle which moves with light velocity, in harmony with Einstein's photon hypothesis. Einstein[7] demonstrated convincingly that Planck's radiation law demands complete symmetry of the emission and absorption mechanism, i.e., something that is unattainable on the basis of the wave equation (even if we operate with the sum of advanced and retarded potential, as suggested by Wheeler and Feynman),[8] while the propagation mechanism derived from the lattice structure of space completely satisfies Einstein's demand.

To this must be added the failure of the wave equation in the face of Bohr's stationary electron orbits. The wave equation permits signal propagation, but is obviously too generous in its operation. *Any* time-dependent disturbance is inevitably propagated with light velocity. And yet we cannot doubt that in atomic dimensions periodic phenomena take place which are *not* connected with any radiation. In our space structure this behavior is entirely natural. The propagation of signals is not an inherent property of a positive definite space structure. The propagation of a signal can occur only under exceptional conditions, namely, if it so happens that on two corresponding ridge surfaces the function $\varphi(x_i)$ assumes the same value in corresponding points. Nor is this rigid hedge hopping restricted to a scalar function. The same happens with the propagation of a vector or tensor field. It is entirely accidental that the condition of radiation is fulfilled by the equality of the tensor components along two corresponding ridge surfaces. But only then will a photon emission take place.

In fact, the phenomenon of signal propagation is even more restricted. The geometry of the ridge surfaces

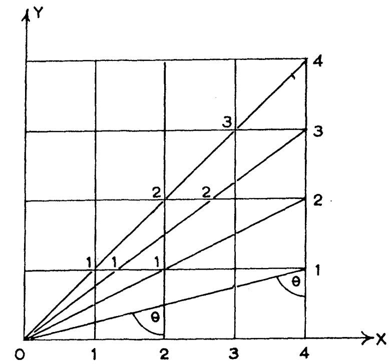

FIG. 2. Macroscopic distance between two lattice points.

cannot make $\partial_n\varphi(x_i)$ exactly zero, but only very small. Hence the condition that $\partial_n\varphi$ shall be the same in the corresponding lattice points x_i and x_i' has to be demanded as an added condition, which overdetermines the problem. We can expect that only a very special type of signals can be propagated, although macroscopically the interaction of a large number of such signals might still give the impression that an arbitrary signal has been transmitted. It is not impossible that the restricted nature of transmittable signals has the consequence that the energy carried by the photon must obey the law $E=h\nu$.

III. RELATIVISTIC KINEMATICS

If the space structure discussed here seems to agree with certain fundamental facts of quantum radiation, is this enough to claim that a positive definite Riemannian space is able to imitate in macroscopic relations the behavior of a Minkowskian space? The behavior of field equations is only *one* part of the picture. Of even more decisive importance are the manifold consequences deduced from special relativity concerning the kinematic and dynamic properties of particles moving according to a certain law of force. All these properties are derived from a Minkowskian type of line element. Can we explain these phenonema on the basis of a space structure which is so completely different? The postulates (1) and (2) exclude the existence of zero lines and yet it is an experimentally proven fact that the apparent lifetime of fast moving particles (mesons) follows exactly the law which makes the four-dimensional distance vanish along the light cone.

In order to answer this question, we will restrict ourselves to two dimensions and operate with the coordinates

$$\begin{aligned} x&=x_4+x_1\\ y&=x_4-x_1. \end{aligned} \tag{7}$$

In this reference system (see Fig. 2) the high ridges become horizontal and vertical lines and in particular the horizontal ridges belong to large values of g_{11}, the vertical ridges to large values of g_{22}.

We will now assume that the Euclidean values

[7] A. Einstein, Z. Physik **18**, 121 (1917).
[8] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **21**, 425 (1949).

$g_{11}=g_{22}=1$ of the plane hold in the majority of points, namely, at all points outside the ridges, and that these values are *very small* in comparison to the large values of the g_{ii} assumed on the ridges. Consequently, any geodetic path which proceeds without any crossing of ridges, is of negligible length if compared to a geodetic path which has to cross one or more ridges. This means that a point which moves either horizontally or vertically covers a negligibly small distance (in the horizontal motion the high ridges of g_{22} are crossed but these contributions are canceled out on account of $dy=0$; similarly, in the vertical motion the high ridges of g_{11} cancel out on account of $dx=0$). Hence any path composed of horizontal and vertical bits is of negligible length; thus, in effect, the zero lines are practically restored. [This does not contradict the metrical postulate (2) which has to hold only in sufficiently small regions. It is clear that on a Riemannian surface generated by a high mountain peak growing out of a flat plane a geodesic crossing of the mountain may be much longer than the sum of two lines drawn in the plane between the two endpoints of the path.]

Let us now move in any other direction. In Fig. 2 the origin of our reference system is connected with the points $(x,1)$, $(x,2)$, $(x,3)$, $\cdots$. We notice that, as we move along these lines, we cross 1, 2, 3, $\cdots$ of the g_{11} ridges. The total length s of the path is thus simply y multiplied by the contribution of a single crossing, since all crossings contribute the same amount to s. This contribution depends on the angle of incidence θ at which the crossing of the ridge takes place. If we characterize this angle by its tangent

$$\tan\theta=x/y\,, \tag{8}$$

we obtain as contribution of the horizontal ridges the expression

$$s=yf(x/y) \tag{9}$$

Similar is the contribution of the vertical ridges, whose number is x throughout (since every straight line crosses all the vertical ridges). Here, however, the angle of incidence is $\pi/2-\theta$ and thus the contribution of these ridges becomes $xf(y/x)$. Hence, the total length of the geodesic between (0,0) and (x,y) becomes

$$s=yf(x/y)+xf(y/x)\,. \tag{10}$$

Now we cannot make any statement concerning the form of the function $f(x/y)$ without constructing an actual model for the metric of the ridges. Let us assume, however, that we have the right to demand that s^2 should become a *quadratic form* of x, y. In that case our choice is strongly restricted. One possible solution is

$$f(x/y)=C/2(x/y)^{1/2}\,, \tag{11}$$

which yields the metric

$$s=C(xy)^{1/2}=C(x_4{}^2-x_1{}^2)^{1/2}\,. \tag{12}$$

This is exactly the Minkowskian distance expression of the plane (x_1,x_4). We have thus demonstrated that our space structure can be harmonized with the requirements of the kinematics and dynamics of special relativity. Moreover, this distance remains invariant under the same transformation which leaves the macroscopic manifestations of the potential equation invariant. The same transformation law can thus be established for the left side and the right side of the equation of motion of a particle.

Another possible solution for the function $f(x,y)$ is given by

$$f\left(\frac{x}{y}\right)=\frac{x^2}{y^2}\frac{C}{\left[1+\left(\frac{x}{y}\right)^2\right]^{\frac{1}{2}}}=C\frac{x^2}{y(x^2+y^2)^{\frac{1}{2}}} \tag{13}$$

which yields for the distance s the expression

$$s=C(x^2+y^2)^{1/2}\,. \tag{14}$$

Here the macroscopic metric *reproduces* the microscopic metric, except for the magnification factor C. We have to assume that both solutions are realized in the physical universe, the first in the three cross sections x_4, $x_\alpha(\alpha=1,2,3)$, the second in the three cross sections x_α, x_β. In the latter case, the distance expression remains invariant with respect to the ordinary rotations of the coordinates; in the former case, with respect to the two-dimensional Lorentz rotations. The resulting group of linear transformations can be described as the totality of transformations which leave the Minkowskian line element invariant.

The constant C of the macroscopic distance expression has to be considered as *very large*, otherwise the contribution of the lengths between the ridges would not be negligible. Originally the motivation for introducing the lattice structure of the world metric was that we demanded a solution of the cosmological equations

$$R_{ik}+\lambda g_{ik}=0\,, \tag{15}$$

where λ is very large with periodic boundary conditions. This would indicate that λ should become of the order of magnitude of unity if the lattice constant σ (the length of the edge of the elementary cell) is chosen as the unit of length. But the fact that we have treated the geometry of the elementary cells as essentially flat (except for the high ridges) indicates that λ must become very *small* if expressed in absolute units (i.e., if we measure lengths in units in which $\sigma=1$).

Under these circumstances we have *two* fundamental lengths entering the theory, viz., the lattice constant σ and the length ρ established by the cosmological constant λ, which has the dimension of a reciprocal length square

$$\lambda=1/\rho^2\,. \tag{16}$$

If the three universal constants $\hbar$, c and the κ in Einstein's matter equation are equated to 1,

$$\hbar = c = \kappa = 1 \tag{17}$$

(which means that Newton's gravitational constant k becomes $1/8\pi$), we obtain the symbolic equation

$$1\ \mathrm{cm} = 1.23 \times 10^{32}, \tag{18}$$

and if we consider the absolute unit of length thus defined as the lattice constant σ, we obtain

$$\sigma = 0.81 \times 10^{-32}\ \mathrm{cm}. \tag{19}$$

According to Treder[9] Planck himself introduced a length of this order of magnitude as a natural unit of length in his celebrated "Lectures on the Theory of Heat Radiation." On the other hand, Heisenberg[10] and, independently, March[11] introduced a fundamental length into quantum theory which is of the order of magnitude 10^{-14} cm. If we identify this length with ρ, we obtain for the cosmological constant the value

$$\lambda = 10^{28}\ \mathrm{cm}^{-2}. \tag{20}$$

[9] H. Treder, Forthschritte der Physik **11**, 81 (1963).
[10] W. Heisenberg, Z. Physik **101**, 533 (1936).
[11] A. March, Z. Physik **105**, 620 (1937).

It is possible that the two lengths σ and ρ are in fact but two aspects of the same length, inasmuch as the length σ is measured microscopically (without crossing any ridges) while ρ is the macroscopic measure of the same length, i.e., the previous length, but multiplied by the magnification factor C. In that case, C would appear as of the order of magnitude 10^{18}. (It is well to remember that the terms "macroscopic" and "microscopic" are of relative significance. What we have called "macroscopic," is physically still submicroscopic, i.e., immeasurably small, while "microscopic" means sub-submicroscopic, i.e., obtainable only by speculative extrapolation, on the basis of Einstein's metaphysical principle that nature is fundamentally reasonable.)

In these discussions we have paid no attention to the question of in what way the metrical properties thus established are deducible from the field equations of the quadratic action principle (which here becomes a genuine minimum principle). Our aim was merely to explore the conclusions we can draw from the possibility that a genuinely Riemannian space of fourfold lattice structure may in macroscopic relations simulate the behavior of a Minkowskian space—the first faltering steps in a new land which is full of intoxicating possibilities.